RELATION between DUAL COMPLEXES and NON-ARCHIMEDEAN GEOMETRY

References

survey Nicaise, 2014, Berkovich skeleta and birational geometry

Mustata-Nicaise, 2015
Weight functions on NA analytic spaces and Kontsevich-Soibelman skeleton

Nicaise-Xu, 2016
The essential skeleton of a degeneration of alg. varieties

de Fernex-Kollár-Xu, 2017
The dual complex of singularities

App 1 Brown-Mazzon, 2019
The essential skeleton of a product of degenerations

Nicaise-Xu-Yu, 2019, The NA SYZ fibration

App 2 Y. Li, 2020 (arxiv) Metric SYZ conjecture and NA geometry

Mazzon-Pille-Schneider, 2021 (arxiv)
Toric geometry and integral affine structures in NA mirror symmetry

Setting
- $\mathcal{C}$ smooth algebraic curve, $0 \in \mathcal{C}$, over $\mathbb{C}$

  $C = \mathcal{C} \setminus \{0\}$

  $\hat{\mathcal{O}}_{\mathcal{C},0} \simeq \mathbb{C}[[t]] =: R$, $K := \mathbb{C}((t))$

  X/C smooth algebraic variety over C

  X_K base change to K

- $\mathcal{X}/\mathcal{C}$ $\overset{\text{dlt}}{\text{model}}$ of X/C

  if $\mathcal{X}_C \simeq X$

  $\mathcal{X}$ normal projective

  $(\mathcal{X}, \mathcal{X}_{0,\mathrm{red}})$ is dlt

Non-archimedean - Part I

Idea: $\underline{\underline{X_K^{an} \supset Sk(\mathcal{X})}} \simeq \underline{D(\mathcal{X}_0)}$ where $(\mathcal{X}, \mathcal{X}_0)$ snc pair

$\mathcal{X}$ model of X_K

1) $X_K^{an} := \left\{ \begin{array}{l} x = (\xi_x, v_x) \\ \xi_x \in X_K \text{ point} \\ v_x : \underset{\substack{\text{residue} \\ \text{field of } X_K \text{ at } \xi_x}}{K(\xi_x)^\times} \longrightarrow \mathbb{R} \text{ valuation} \\ \text{such that } v_x \text{ extends } v_K = \mathrm{ord}_t \end{array} \right\}$

$$\begin{aligned} \iota : X^{an} &\longrightarrow X \quad \text{forgetful map} \\ x &\longmapsto \xi_x \end{aligned}$$

$\imath : X_K^{an} \longrightarrow X_K \qquad x = (\xi_x, v_x) \longmapsto \xi_x$

topology: weakest such that

$$v(f) : \imath^{-1}(U) \longrightarrow \mathbb{R} \qquad \text{is continuous}$$
$$x \longmapsto v_x(f)$$

for any $f \in \mathcal{O}_X(U)$, $U \subseteq X_K$ Zariski open

$$X_K^{an} \supset X_K^{bir}$$
$$\| $$
$$\{ v_x : K(X)^\times \to \mathbb{R} \} = \imath^{-1}(\eta_X)$$

extending v_K $\qquad$ $\uparrow$ generic pt of X_K

birational invariant of X_K

2) $D(\mathcal{X}_0) \hookrightarrow X_K^{bir} \subset X_K^{an} \qquad \mathcal{X}_0 = \sum E_i$

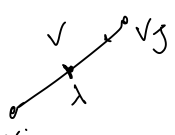
$$v_i \longmapsto \mathrm{ord}_{E_i}$$
$$\updownarrow$$
$$E_i$$

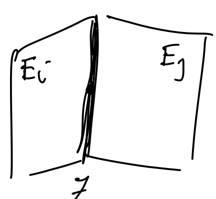
$v \in \langle v_i, v_j \rangle \longmapsto$ v_{min} minimal valuation on $\mathcal{O}_{\mathcal{X}, \eta_Z}$

z_i, z_j local equations of E_i, E_j at η_Z

s.t. $v_{min}(z_i) = \lambda$

$v_{min}(z_j) = 1 - \lambda$





$t = u \cdot z_i z_j$ at η_Z

$Z \subseteq E_i \cap E_j \subseteq \mathcal{X}_0$

$1 = v_K(t) = v_{min}(z_i) + v_{min}(z_j)$

$$D(\mathcal{X}_0) \simeq \underset{\substack{\text{skeleton} \\ \text{of } \mathcal{X}}}{Sk(\mathcal{X})} \hookrightarrow X_K^{bir} \subset X_K^{an}$$

3) $\rho_{\mathcal{X}} : X_K^{an} \longrightarrow Sk(\mathcal{X})$

retraction

- center of $x \in X_K^{an}$

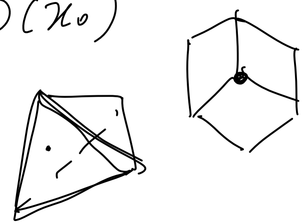
$x = (\xi_x, v_x)$

$$\begin{tikzcd}
\operatorname{Spec} K(\xi_x) \arrow[r, "\xi_x"] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} K(\xi_x)^\circ \arrow[r] & \mathcal{X}
\end{tikzcd}$$

$K(\xi_x)^\circ$ = valuation ring of $K(\xi_x)$ wrt v_x

$$m_x \longmapsto c_{\mathcal{X}}(v) \in \mathcal{X}_0$$

- $v \in X_K^{an}$ $\qquad \rho_{\mathcal{X}}(v) \in Sk(\mathcal{X}) \simeq D(\mathcal{X}_0)$



$v \rightsquigarrow c_{\mathcal{X}}(v) \in \mathcal{X}_0$

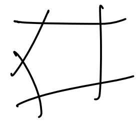
$\in$ minimal stratum Z of $\mathcal{X}_0$ $= \bigcap_{i=1}^{r} E_i$ $\Rightarrow \rho_{\mathcal{X}}(v) \in \sigma_Z$

$\rho_{\mathcal{X}}(v)(z_i) = v(z_i)$

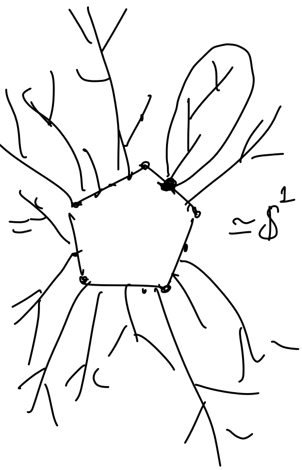
local equations of E_i at $c_{\mathcal{X}}(v)$

4) $X_k^{an} \xrightarrow{\rho_{\mathcal{X}}} \varprojlim_{\mathcal{X} \text{ snc}} Sk(\mathcal{X})$ is homeomorphism

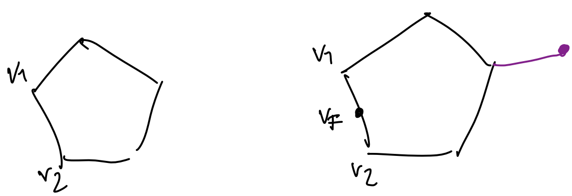
<u>Example</u>: E_k elliptic curve

$\mathcal{E}$ snc model s.t. $\mathcal{E}_0 =$ 

E_k^{an}

$D(\mathcal{E}_0) \simeq Sk(\mathcal{E}) =$  $\simeq \mathbb{S}^1$

$\mathcal{E}_0$



$$\rho_{\mathcal{E}} : E_k^{an} \longrightarrow Sk(\mathcal{E}) \simeq \mathbb{S}^1$$

Essential skeleton

1) [Nicaise-Xu, Thm 2.3.6]:

X/C has a good minimal dlt model $\iff$ K_X semi-ample

$\|$

$\begin{cases} \mathcal{X} \text{ dlt model of } X/C \\ \mathcal{X} \text{ is } \mathbb{Q}\text{-factorial} \\ K_{\mathcal{X}} + \mathcal{X}_{0,red} \text{ semiample over } \mathcal{C} \end{cases}$

2) Any two good minimal dlt models $\mathcal{X}_1, \mathcal{X}_2$ are crepant birational

2) $\Rightarrow$ by [Proposition 11, dF-K-X]

$Sk(\mathcal{X}_1) \simeq Sk(\mathcal{X}_2)$

PL-homeomorphism

defn: For X/C with K_X semi-ample, we define

$Sk(X) := Sk(\mathcal{X}_{min})$

w/ $\mathcal{X}_{min}$ any good minimal dlt model of X

$$X_K^{an} \supset X_K^{bir} \supset Sk(\mathcal{X}) \qquad \mathcal{X} \text{ snc}$$

$$\cup$$

$Sk(X)$ essential skeleton

(birational invariant of X)

Non-archimedean - Part II

K_X semi-ample
$\omega \in H^0(X, K_X^{\otimes m})$ non-zero

$$\mathrm{wt}_\omega : X_K^{an} \longrightarrow \mathbb{R} \cup \{+\infty\} \quad \text{weight function}$$

- $v_E = (\mathcal{X}, E)$ $\quad$ $\mathrm{wt}_\omega(v_E) = v_E\left(\mathrm{div}_{\mathcal{X}}(\omega) + m\, \mathcal{X}_{0,red}\right)$

  $v \in Sk(\mathcal{X})$

  see ω as a rational section of $K_{\mathcal{X}}$ in $\mathcal{X}$

  [Mustata - Nicaise, Prop 4.3.4]: wt_ω well-defined

- $v_E = (\mathcal{X}, E)$ $\quad$ $\mathrm{wt}_\omega(v_E) \geq \mathrm{wt}_\omega(\rho_{\mathcal{Y}}(v_E))$

  $\mathcal{Y}$ snc model $\quad$ with equality iff $v_E \in Sk(\mathcal{Y})$

  $$\rho_{\mathcal{Y}} : X_K^{an} \to Sk(\mathcal{Y})$$
  $$v_E \mapsto \rho_{\mathcal{Y}}(v_E)$$

- $v \in X_K^{an}$ $\quad$ $\mathrm{wt}_\omega(v) := \sup_{\mathcal{X}} \left\{ \mathrm{wt}_\omega(\rho_{\mathcal{X}}(v)) \right\}$

defn: $Sk(X, \omega) = \{ v \in X_K^{an} \mid \mathrm{wt}_\omega(v) \text{ is minimal} \}$

Kontsevich-Soibelman skeleton

$\cap|$

$Sk(\mathcal{X})$ any $\mathcal{X}$ snc model defn: $\mathrm{Sk}^{ess}(X) := \bigcup_{\substack{\omega \text{ non-zero} \\ \text{regular pluricanonical form}}} \mathrm{Sk}(X,\omega) \subset \mathrm{Sk}(\mathcal{X}) \subset X_K^{\mathrm{ber}}$

[Theorem 3.3.3, Nicaise-Xu]:

If K_X semiample

$$\boxed{\mathrm{Sk}(X) = \mathrm{Sk}^{ess}(X)}$$

$$\mathrm{Sk}(X) = \mathrm{Sk}(\mathcal{X}_{min}), \qquad \mathrm{Sk}^{ess}(X) = \bigcup_{\omega} \mathrm{Sk}(X,\omega)$$

$\mathcal{X}_{min}$ good minimal dlt

Applications

I) Dual complex of dlt degenerations

S K3 surface, $K_S \sim \mathcal{O}_S$

$\mathrm{Hilb}^n(S) \sim S^{(n)} = S^n/\sigma_n$

$$\mathrm{Sk}^{ess}(\mathrm{Hilb}^n(S)) = \mathrm{Sk}^{ess}(S^n/\sigma_n) \simeq \mathrm{Sk}^{ess}(S)^n/\sigma_n$$

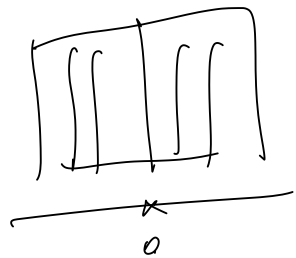
[Brown-Mazzon]

PL-homeomorphism

$$Sk^{ess}(\mathrm{Hilb}^n(S)) \simeq Sk^{ess}(S)^n / \sigma_n$$

$$= \begin{cases} pt & \text{if } Sk^{ess}(S) = pt & \text{type I} \\ \Delta_n & \text{if } Sk^{ess}(S) = [0,1] & \text{type II} \\ \mathbb{CP}^n & \text{if } Sk^{ess}(S) = S^2 \simeq \mathbb{CP}^1 & \text{type III} \end{cases}$$

$S/C \qquad C = \mathbb{C} \setminus \{0\}$

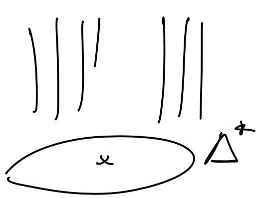


Rmk: [Kollár–Laza–Saccà–Voisin] 1 degeneration of HK manifolds

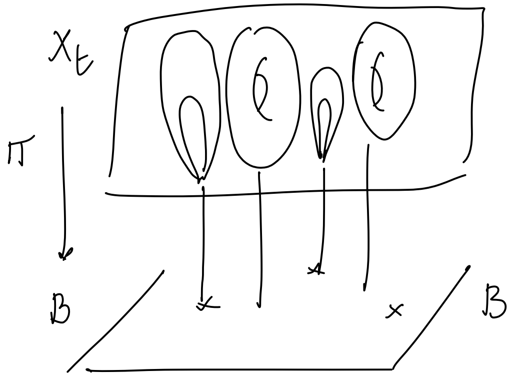
II) SYZ conjecture

$X/\Delta^* \subset \mathbb{C}$ max degenerate family of CY varieties of dim n

$\mathcal{X}/\Delta \qquad \dim_{\mathbb{R}} D(\mathcal{X}_0) = \dim_{\mathbb{C}} X_t = n$



SYZ conj: For $0 < |t| \ll 1$, X_t admits a map $\pi : X_t \to B$ which is a special Lagrangian torus fibration away from a locus of codim ≥ 2 in B



$\dim_{\mathbb{C}} X_t = n$

$\dim_{\mathbb{R}} B = n$

[Nicaise-Xu-Yu, Thm 6.1]

Let $\mathcal{X}$ be a good minimal dlt of X_K

Then $\rho_{\mathcal{X}} : X_K^{an} \longrightarrow Sk(X)$ is an affinoid torus fibration away from a locus of codim ≥ 2 in $Sk(X)$.

$z_1, \dots, z_n$

$$\begin{array}{ccccc} X_K^{an} & & (\mathbb{G}_m^n)^{an} & & v \\ \downarrow \rho_{\mathcal{X}} & \text{locally isomorphic} & \downarrow \text{trop} & & \downarrow \\ Sk(X) & & \mathbb{R}^n & & (v(z_i)) \end{array}$$

Rmk: Yang Li:

Up to a NA conjecture about f_x,

SYZ conjecture hold on the generic region of X_t (metrically)

work in progress

Jonsson - M. - McCleerey:

NA conjecture holds for CY hypersurface in $\mathbb{P}^n$